Tensor product in symmetric function spaces

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Abstract

A concept of multiplicator of symmetric function space concerning to projective tensor product is introduced and studied. This allows us to obtain some concrete results. In particular, the well-known theorem of R.O'Neil about the boundedness of tensor product in the Lorentz spaces L_{pq} is discussed.

0. *Introduction*

Let $x = x(s)$ and $y = y(t)$ are measurable functions on $I = [0, 1]$. We define following bilinear operator:

$$B(x, y)(s, t) = (x \otimes y)(s, t) = x(s)y(t), (s, t) \in I \times I$$

If X, Y, Z are symmetric function spaces, then the boundedness of B from $X \times Y$ into $Z(I \times I)$ is equivalent to the continuity of the embedding

$$X \otimes Y \subset Z$$

where $X \otimes Y$ denotes the projective tensor product of spaces X and Y [1, p.51].

The importance of study of operator B follows mainly from two facts : this operator has various applications in the theory of symmetric spaces (see, for example, [2,p.169-171], [3], [4]) and the problem of boundedness of tensor product is equivalent to the stability problem of integral operators ([1, p. 50], [5]).

The problem of boundedness of the operator B in certain families of Banach spaces (for example, Lorentz, Marcinkiewicz and Orlicz spaces) has attracted the attention of number of authors (see [1], [6 - 11]). In particular, R.O'Neil proved following theorem about the boundedness of tensor product in the spaces L_{pq} [9].

Let us remind that the space $L_{pq}(1 < p < \infty, 1 \le q \le \infty)$ consists of all measurable on I functions $x = x(t)$ for which

$$||x||_{pq} = \begin{cases} \left\{ \int\limits_0^1 \left(x^*(t)t^{1/p} \right)^q dt/t \right\}^{1/q}, & \text{if } 1 \le q < \infty \\ \sup\limits_{0<t\le 1} \left(x^*(t)t^{1/p} \right), & \text{if } q = \infty \end{cases}$$

is finite. Although the functional $\|x\|_{pq}$ is not subadditive, however it is equivalent to the norm $||x||'_{pq} = ||x^{**}||_{pq}$. where $x^{**}(t) = 1/t \int_0^t x^*(s)\, ds$ and by $x^*(s)$ is denoted the left continuous non-increasing rearrangement of $|x(s)|$.

Let us notice that $L_{pr} \subset L_{pq}(r \le q), L_{pp} = L_p$ and usual norm of L_p $||x||_p = ||x||_{pp}$. These spaces arise naturally in interpolation theory (see, for example, [12]).

Theorem (O'Neil). Let $1 < p < \infty, 1 \leq q, r, s \leq \infty$. The boundedness of the operator B from $L_{pr} \times L_{pq}$ into $L_{ps}(I \times I)$ is equivalent to the conditions :

$$1) \max(q, r) \leq s$$

and

$$2) \frac{1}{p} + \frac{1}{s} \leq \frac{1}{q} + \frac{1}{r}$$

.

We consider following problems connected with the exactness of this theorem :

a) Let us fix $1 < p < \infty, 1 \leq r \leq s \leq \infty$ and q^* is the maximum of all $q \geq 1$ for which the conditions of theorem are satisfied (if those exist). Is the space L_{pq^*} the largest among symmetric spaces E such that

$$B : L_{pr} \times E \to L_{ps}(I \times I) \text{ ?}$$

b) Let us $1 < p < \infty, 1 \leq r \leq q \leq \infty$ and s^* is the minimum of all $s \geq 1$ for which the conditions of theorem are satisfied (if those exist). Is the space L_{ps^*} the smallest among symmetric spaces E such that

$$B : L_{pr} \times L_{pq} \to E(I \times I) \text{ ?}$$

The answers on these questions are positive, of course, in case $r \leq p$, since B bounded from $L_{pr} \times L_{pr}$ into $L_{pr}(I \times I)\,(r \leq p)$ and from $L_p \times L_{pq}$ into $L_{pq}\,(p \leq q)$ ([9],[11]).

In this paper we shall study tensor product in general symmetric spaces. It will allow us to solve in part the problems a) and b).

The paper is organized in the following manner. At first, we introduce a concept of the multiplicator $M(E)$ of a symmetric space E concerning to tensor product. The relations defining the fundamental function and the norm of dilation operator in $M(E)$ will be obtained. Moreover, the upper and lower estimates of $M(E)$ allow to find it in some cases. In particular, we shall get the positive answer on the question a) in case $p < r = s$ (see the corollary 1.12). Then we shall consider a quantity connected with the boundedness of tensor product in symmetric spaces.

Finally we study the boundedness of operator B in the spaces L_{pq} using the family of Lorentz-Zygmund spaces $L_{pq}(\log L)^{\alpha}$ that involve, in particular, spaces L_{pq}. The main result — theorem 2.1 — is similar to O'Neil theorem. First of all, it allows to define more exactly values of B on $L_{pr} \times L_{pq}$ for $p < r \leq q$, $q^{-1} + r^{-1} - p^{-1} \geq 0$. Namely, we shall find (see corollary 2.7) the symmetric space $E = E(p, r, q) \subset L_{ps}$, $E \neq L_{ps}$, $s^{-1} = q^{-1} + r^{-1} - p^{-1}$ such that

$$B : L_{pr} \times L_{pq} \to E(p, r, q)\,(I \times I)$$

The last means that the answer on the problem b) is negative in this case. Besides, O'Neil theorem does not describe the set $B\,(\,L_{pr},\,L_{pq}\,)$, if $r^{-1} + q^{-1} - p^{-1} < 0$ (a family L_{pq} is too narrow for study of this problem). Using the spaces $L_{pq}(\log L)^{\alpha}$ we solve this problem to within any positive power of logarithm (see theorems 2.1 and 2.4). The complex method of interpolation is used here (the definitions of interpolation theory see in [12] or [13]).

1. *Thr multiplicator of symmetric space concerning to tensor product*

If $z = z(w)$ is a measurable function on I or $I \times I$, μ is the usual Lebesgue measure, then the distribution function of $|z(s)|$ is $n_z(\tau) = \mu\{w : |z(w)| > \tau\}$ $(\tau > 0)$. The function $x(t)$ is equimeasurable with $y(t)$ if $n_x(\tau) = n_y(\tau)$ $(\tau > 0)$.

Let us remind that a Banach space E of measurable functions defined on I is said to be symmetric if these conditions are satisfied: a) $y \in E$ and $|x(t)| \leq |y(t)|$ imply that $x \in E$ and $||x|| \leq ||y||$; b) if $y \in E$ and the function $x(t)$ is equimeasurable with $y(t)$, then $x \in E$ and $||x|| = ||y||$.

Let E be a symmetric space on $I = [0, 1]$. We shall denote by $M(E)$ the set of all measurable functions $x = x(s)$ $(s \in I)$ for which $x \otimes y \in E(I \times I)$ with arbitrary $y \in E$. Then $M(E)$ is symmetric space on I concerning to the norm

$$||x|| = sup\{ ||x \otimes y||_{E(I\times I)} : ||y||_E \leq 1 \}$$

Obvios, $M(E) \subset E$.

Example 1.1. Let us find $M(E)$, if E is Lorentz space $\Lambda(\phi)$, where positive concave function $\phi(u)$ increases on $(0, 1]$. This space consists of all measurable functions $x = x(s)$ for which

$$||x||_{\Lambda(\phi)} = \int_0^1 x^*(s)\, d\phi(s) < \infty$$

In particular, $L_{p1} = \Lambda(t^{1/p})$ $(1 < p < \infty)$.

For arbitrary $e \subset I$, $x \in \Lambda(\phi)$ functions $\chi_e \otimes x$ and $\sigma_{\mu(e)}x$ are equimeasurable, where $\chi_e(s) = 1\,(s \in e)$, $\chi_e(s) = 0\,(s \notin e)$ and the dilation operator $\sigma_t y(u) = y(u/t)\chi_{[0,1]}(u/t)$ $(t > 0)$. The rearrangements of equimeasurable functions are equal a.e.. Therefore,

$$||\chi_e \otimes x||_{\Lambda(\phi)} = \int_0^1 (\sigma_{\mu(e)}\, x)^*(u)\, d\phi(u) = \int_0^{\mu(e)} x^*(u/\mu(e))\, d\phi(u) = \int_0^1 x^*(v)\, d\phi(\mu(e)v)$$

Since $||\chi_e||_{\Lambda(\phi)} = \phi(\mu(e))$, then we have in view of [12, p.151]

$$||x||_{M(\Lambda(\phi))} \leq 2 \sup_{0<t\leq 1} \int_0^1 x^*(v)\, d\left[\frac{\phi(tv)}{\phi(t)}\right].$$

Introduce the notation:

$$\mathcal{M}_\phi(v) = \sup\left\{ \frac{\phi(tv)}{\phi(t)},\ 0 < t \leq \min(1, 1/v) \right\}.$$

Using last inequality, we obtain

$$\Lambda(\mathcal{M}_\phi) \subset M(\Lambda(\phi))$$

Assume in addition that

$$\mathcal{M}_\phi(v) = \lim_{t\to t_0} \frac{\phi(tv)}{\phi(t)}, \tag{1}$$

where $t_0 \in [0, 1]$ does not depend on $v \in [0, 1]$. In this case

$$||x||_{\Lambda(\mathcal{M}_\phi)} = \lim_{t\to t_o} \int_0^1 x^*(v)\, d\left[\frac{\phi(tv)}{\phi(t)}\right] = \lim_{t\to t_0} \frac{||\chi_{(0,t)} \otimes x||_{\Lambda(\phi)}}{\phi(t)} \leq ||x||_{M(\Lambda(\phi))}.$$

Consequently, if (1) holds, then

$$M(\,\Lambda(\,\phi\,)\,)\;=\;\Lambda(\,\mathcal{M}_\phi\,).$$

Let again E be arbitrary symmetric space on I. We shall obtain the relations connecting the fundamental functions and the norms of dilation operators of spaces E and $M(E)$.

In the first place,

$$\phi_{M(E)}(t)\;=\;||\sigma_t||_{E\to E},\;\;0<t\leq 1, \tag{2}$$

where $\phi_X(t)$ denotes the fundamental function of symmetric space X: $\phi_X(t)=||\chi_{(0,t)}||_X$. Indeed, since the functions $\chi_{(0,t)}\otimes y$ and $\sigma_t y$ ($t\in[0,1]$) are equimeasurable, then

$$\phi_{M(E)}(t)\;=\;\sup_{||y||\leq 1}||\chi_{(0,t)}\otimes y||_E\;=\;\sup_{||y||\leq 1}||\sigma_t y||_E\;=\;||\sigma_t||_{E\to E}.$$

Theorem 1.2. For arbitrary symmetric space E on I :

$$||\sigma_t||_{M(E)\to M(E)}\;=\;||\sigma_t||_{E\to E},\;\;0<t\leq 1 \tag{3}$$

$$||\sigma_{1/t}\,||_{E\to E}{}^{-1}\;\leq\;||\sigma_t||_{M(E)\to M(E)}\;\leq\;||\sigma_t||_{E\to E}\,,\;\;t>1 \tag{4}$$

Proof. First of all, if $x(u)$ and $y(v)$ are measurable functions on I and

$$supp\,x\;=\;\{u:\;x(u)\neq 0\}\;\subset\;[0,\min(1,t^{-1})],$$

then for $t,s>0$

$$n_{\sigma_t x\otimes y}(s)=\int_0^1\mu\left\{u:\;|\sigma_t x(u)|\;>\;\frac{s}{|y(v)|}\right\}dv=t\int_0^1\mu\left\{u:\;|x(u)|\;>\;\frac{s}{|y(v)|}\right\}dv=t\;\;n_{x\otimes y}(s).$$

Consequently,

$$(\,\sigma_t x\otimes y\,)^*\,(s)\;=\;\sigma_t(x\otimes y)^*\,(s)\;\;\;0\leq s\leq 1,$$

and thus

$$||\sigma_t x\otimes y||_E\;=\;||\sigma_t(x\otimes y)^*||_E\;\leq\;||\sigma_t||_{E\to E}\,||x\otimes y||_E \tag{5}$$

If $0<t\leq 1$, then we get from (5) by definition of norm in $M(E)$

$$||\sigma_t||_{M(E)\to M(E)}\;\leq\;||\sigma_t||_{E\to E} \tag{6}$$

In case $t>1$ we define the function $\tilde{x}(u)=x(u)\chi_{(0,1/t]}(u)$. For it (5) is satisfied and, besides, the functions $\sigma_t\tilde{x}$ and $\sigma_t x$ are equimeasurable. Therefore,

$$||\sigma_t x\otimes y||_E\;\;=\;||\sigma_t\tilde{x}\otimes y||_E\;\;\leq\;||\sigma_t||_{E\to E}\,||\tilde{x}\otimes y||_E\;\;\leq\;||\sigma_t||_{E\to E}\,||x\otimes y||_E,$$

and thus (6) is satisfied for $t>1$.

If $0<t\leq 1$, then in view (2) and by the quality $||\chi_{(0,1)}||_{M(E)}\;=\;1$

$$||\sigma_t||_{E\to E}\;\;=\;\phi_{M(E)}(t)\;=||\sigma_t\chi_{(0,1)}||_{M(E)}\;\;\leq\;||\sigma_t||_{M(E)\to M(E)}$$

Using the submultiplicativity of function $f(t) = ||\sigma_t||_{M(E)\to M(E)}$ $(t > 0)$ and the inequality (6) we get in case $t > 1$:

$$||\sigma_t||_{M(E)\to M(E)} \;\geq\; ||\sigma_{1/t}||_{M(E)\to M(E)}^{\;\;-1} \;\geq\; ||\sigma_{1/t}\,||_{E\to E}^{\;\;-1}$$

The theorem is proved.

The Boyd indices of symmetric space E are defined as

$$\alpha_E \;=\; \lim_{t\to 0} \frac{||\sigma_t||_{E\to E}}{\ln t} \;,\; \beta_E \;=\; \lim_{t\to\infty} \frac{||\sigma_t||_{E\to E}}{\ln t} \;.$$

From theorem 1.2 follows

Corollary 1.3. If E is any symmetric space on I, then

$$0 \leq \alpha_E \;=\; \alpha_{M(E)} \;\leq\; \beta_{M(E)} \;\leq\; \beta_E \;\leq\; 1.$$

Remark 1.4. Left-hand side of the inequality (4) cannot be improved, in general. We shall show that there exists a symmetric space E on I such that

$$||\sigma_t||_{M(E)\to M(E)}\,||\sigma_{1/t}\,||_{E\to E} \;=\; 1\,(t \geq 1) \tag{7}$$

and

$$\lim_{t\to\infty} \frac{||\sigma_t||_{M(E)\to M(E)}}{||\sigma_t||_{E\to E}} \;=\; 0 \tag{8}$$

Let $\phi_\alpha(s) = \; s^\alpha \ln^{-1}(C/s)$, $E = \Lambda(\,\phi_\alpha\,)\,(0 < \alpha < 1)$. If $C > \exp\{1/(1-\alpha)\}$, then the function ϕ_α is concave on $(0, 1]$. The function $f_t(s) = \; \ln(C/s)\; \ln^{-1}[C/(st)]$ $(0 < s \leq \min(1, 1/t)\,)$ decreases, if $0 < t \leq 1$, and increases, if $t > 1$. Therefore, for $0 < t \leq 1$

$$\mathcal{M}_{\phi_\alpha}(t) = \; t^\alpha \lim_{s\to 0+} f_t(s) = \; t^\alpha.$$

Hence, in particular, it follows that the condition (1) is fulfilled for $\phi_\alpha(s)$ and consequently as it was showed in example 1.1

$$M(\,\Lambda(\,\phi_\alpha\,)\,) \;=\; \Lambda(\,\mathcal{M}_{\phi_\alpha}\,) \;=\; \Lambda(\,t^\alpha\,)$$

In the same time, for $t > 1$

$$\mathcal{M}_{\phi_\alpha}(t) \;=\; t^\alpha f_t(1/t) \;=\; t^\alpha \ln(Ct)\ln^{-1} C$$

Since $||\sigma_t||_{\Lambda(\phi)\to\Lambda(\phi)} \;\; = \;\; \mathcal{M}_\phi(t)$ [12, p. 134], then in our case

$$||\sigma_t||_{E\to E} \;\;=\;\; \mathcal{M}_{\phi_\alpha}(t),\;\; ||\sigma_t||_{M(E)\to M(E)} \;\;=\;\; t^\alpha \;\; (t > 0)$$

As a result we obtain (7) and (8).

Continuing the study of multiplicator of a symmetric space, we shall find its upper and lower estimates in general case.

Theorem 1.5(upper estimate). Suppose that E is a symmetric space on I, $p = 1/\alpha_E$, where α_E is lower Boyd index of space E. Then

$$M(E) \subset L_p$$

and the constant of this embedding does not depend from a space E.

Proof. We consider the nonnegative simple functions of type:

$$x(s) = \sum_{i=1}^{m} \alpha_i \chi_{(\frac{i-1}{m}, \frac{i}{m}]}(s), \ \alpha_i \geq 0, \, m \in \mathbf{N} \tag{9}$$

For any $y \in E$ function $x \otimes y$ is equimeasurable with the function

$$z_{x,y}(t) = \sum_{k=1}^{m} \alpha_k y_k(t),$$

where $y_k \in E$ are disjointly supported functions having all the same distribution function and

$$n_{y_k}(s) = \frac{1}{m} n_y(s), \ s > 0, \, k = 1, 2, .., m \tag{10}$$

We shall show that for every $\epsilon > 0$ there exists $y \in E$ (or y_k, $k = 1, 2, .., m$) for which the equalities (10) hold and

$$||x \otimes y||_E = ||z_{x,y}||_E \geq \frac{1-\epsilon}{1+\epsilon} ||x||_p \, ||y||_E, \tag{11}$$

where x is arbitrary function of type (9) and $p = 1/\alpha_E$.

Indeed, in view of [2, p. 141] (see also [14]) one can choose disjointly supported equimeasurable functions $z_k \in E \, (k = 1, 2, .., m)$ such that for $\alpha_k \geq 0$

$$(1-\epsilon) \left(\sum_{k=1}^{m} {\alpha_k}^p \right)^{1/p} \leq || \sum_{k=1}^{m} \alpha_k z_k ||_E \leq (1+\epsilon) \left(\sum_{k=1}^{m} {\alpha_k}^p \right)^{1/p} \tag{12}$$

Denote

$$y_k = \frac{z_k}{m} \, (k = 1, 2, .., m), \ y = \frac{1}{m} \sum_{k=1}^{m} z_k.$$

These functions satisfy all necessary assumptions, in particular, the equalities (10). We shall prove (11).

From (12) (if we take $\alpha_k = 1$ for $k = 1, 2, .., m$) it follows that

$$||y||_E \leq (1+\epsilon) m^{1/p-1},$$

and therefore

$$|| \sum_{k=1}^{m} \alpha_k y_k ||_E = \frac{1}{m} || \sum_{k=1}^{m} \alpha_k z_k ||_E \geq \frac{1-\epsilon}{m} \left(\sum_{k=1}^{m} {\alpha_k}^p \right)^{1/p} \geq$$

$$\geq \frac{1-\epsilon}{1+\epsilon} m^{-1/p} \, ||y||_E \left(\sum_{k=1}^{m} {\alpha_k}^p \right)^{1/p} = \frac{1-\epsilon}{1+\epsilon} ||x||_p \, ||y||_E.$$

Since $\epsilon > 0$ is arbitrary, then the following inequality can be obtained for functions $x(s)$ of type (9):

$$||x||_{M(E)} \geq ||x||_p \tag{13}$$

Let us extend last inequality on countably-value dyadic functions

$$x(s) = \sum_{k=1}^{\infty} \alpha_k \, \chi_{(2^{-k}, 2^{-k+1}]}(s), \;\; \alpha_k \geq 0.$$

Functions $x_n(s) = x(s)\chi_{(2^{-n},1]}(s)$ belong to the class (9), $x_n(s) \leq x(s)\,(n = 1, 2, ..)$. Consequently,

$$||x||_{M(E)} \geq ||x_n||_{M(E)} \geq ||x_n||_p \; (n = 1, 2, ..).$$

Passing to the limit as $n \to \infty$ we obtain (13).

If now $y \in E$ is arbitrary, then we define the function

$$x_y(s) = \sum_{k=1}^{\infty} y^*(\, 2^{-k+1}\,)\, \chi_{(2^{-k}, 2^{-k+1}]}(s).$$

It is clear that $x_y \leq y^* \leq \sigma_2(x_y)$. Since for any symmetric space X $||\sigma_t||_{X \to X} \leq \max(1, t)$ [12, p. 133], then

$$||y||_{M(E)} = ||y^*||_{M(E)} \geq ||x_y||_{M(E)} \geq ||x_y||_p \geq \frac{1}{2}||y||_p.$$

The theorem is proved.

Theorem 1.6 (lower estimate). For arbitrary symmetric space E on I

$$M(E) \supset \Lambda(\,\psi\,),$$

where $\psi(t) = ||\sigma_t||_{E \to E}$ $(\,0 < t \leq 1\,)$.

Moreover, the constant of embedding does not depend from a space E.

Proof. We fix $y \in E$ and consider the operator $T_y x = x \otimes y$. If $x = \chi_e$, $e \subset [0, 1]$, $\mu(e) = t$,then

$$||T_y \chi_e||_E = ||\sigma_t y||_E \leq ||\sigma_t||_{E \to E}\, ||y||_E = \psi(t)||y||_E.$$

In view of [12, p. 151] the following inequality is true for every $x \in \Lambda(\,\psi\,)$:

$$||x \otimes y||_E = ||T_y x||_E \leq 2||y||_E\, ||x||_{\Lambda(\psi)}.$$

The theorem is proved.

Consider some corollaries of obtained theorems.

Corollary 1.7. The multiplicator $M(E) = L_\infty$ iff the lower Boyd index of symmetric space E $\alpha_E = 0$.

Proof. If $\alpha_E = 0$, then $M(E) = L_\infty$ in view of theorem 1.5. If $\alpha_E > 0$, then by theorem 1.6 $M(E) \supset \Lambda(\psi)$ and $\psi(t) = ||\sigma_t||_{E \to E} \to 0\,(t \to 0+)$. Therefore $\Lambda(\psi) \neq L_\infty$. Since $\Lambda(\psi) \supset L_\infty$ [12,p.124] then, all the more, $M(E) \neq L_\infty$.

Corollary 1.8. If $B: \ E\times E \to E(I\times I)$, then $E\subset L_{1/\alpha_E}$.

If E is a Banach space of measurable functions defined on I, then dual space E' consists of all measurable on I functions $y = y(t)$ for which

$$||y||_{E'} \ = \ \sup\{\int_0^1 x(t)y(t)dt: \ ||x||_E \leq 1\} \ < \ \infty.$$

The norm of symmetric space E is called order subcontinuous if from monotone convergence $x_n \uparrow x$ a.e. $(x_n, x\in E)$ it follows: $||x_n||_E \to ||x||_E$.

Corollary 1.9. Let E be a symmetric space with order subcontinuous norm. If $B: \ E\times E \to E(I\times I)$ and $B: \ E'\times E' \to E'(I\times I)$, then there exists $p\in[1,\infty]$ such that $E = L_p$.

Proof. By corollary 1.8 $E\subset L_{1/\alpha_E}$. For dual spaces the inverse embedding is satisfied:

$$E' \supset (L_{1/\alpha_E})' \ = \ L_{1/1-\alpha_E} \ \supset L_{1/\beta_{E'}} \ \supset L_{1/\alpha_{E'}},$$

since $1-\alpha_E \geq \beta_{E'} \geq \alpha_{E'}$ [12, p. 144].

On the other hand, now in view of corollary 1.8 $E'\subset L_{1/\alpha_{E'}}$ and thus $E' = L_{1/\alpha_{E'}}$.

As $L_\infty \subset E$ and E is the subspace of E'' [15, p.255] we obtain, finally, for $p = 1/(1-\alpha_{E'})$ the equality:

$$E \ = \ E'' \ = \ L_p.$$

If $\phi_E(s)$ is the fundamental function of a symmetric space E, then $\mathcal{M}_{\phi_E}(t) \leq ||\sigma_t||_{E\to E}$. A space E is called space of fundamental type provided

$$||\sigma_t||_{E\to E} \ \leq \ C\mathcal{M}_{\phi_E}(t),$$

where C does not depend on $t>0$. Let us notice that all most important in applications symmetric spaces are spaces of fundamental type.

Corollary 1.10. If E is a symmetric space of fundamental type, then $M(E)\supset \Lambda(\mathcal{M}_{\phi_E})$

Let us remind that a Banach space E is interpolation space between the spaces E_0 and E_1 if $E_0\cap E_1 \subset E \subset E_0 + E_1$ and from boundedness of arbitrary linear operator in E_0 and E_1 follows its boundedness in E. In this case there exists $C>0$ such that

$$||T||_{E\to E} \leq C\max_{i=0,1} ||T||_{E_i\to E_i}$$

where C does not depend on operator T[12].

Theorem 1.11. Let symmetric space E be an interpolation space between the spaces L_p and $L_{p\infty}$ by some $p\in(1,\infty)$. Then $M(E) = L_p$.

Proof. Since $L_p \subset E \subset L_{p\infty}$, then, obvious, there exist $C_1>0$ and $C_2>0$ for which

$$C_1 t^{1/p} \leq \phi_E(t) \leq C_2 t^{1/p} \ (0\leq t\leq 1).$$

Hence, in particular,

$$||\sigma_t||_{E\to E} \ \geq \ C_3 t^{1/p} \ (t>0).$$

On the other hand, by condition of theorem

$$||\sigma_t||_{E\to E} \ \leq C_4 \max\Big(||\sigma_t||_{L_p\to L_p}\,,\ ||\sigma_t||_{L_{p\infty}\to L_{p\infty}}\Big) \ = \ C_4 t^{1/p}.$$

Using obtained inequalities, we get in view of definition of Boyd indices $\alpha_E \ = \ \beta_E \ = \ 1/p$, and therefore by the theorem 1.2 $M(E)\subset L_p$.

For the proof of inverse embedding we fix $y\in L_p$ and consider the linear operator $T_y x = \ x\otimes y$. Then

$$||T_y||_{L_p\to L_p} \ = \ ||y||_p \ \ , \ \ ||T_y||_{L_{p\infty}\to L_{p\infty}} \ \leq \ C_5||y||_p,$$

where $C_5>0$ does not depend from y [11]. Consequently, in view of interpolation property of E

$$||T_y||_{E\to E} \ \leq \ C_6\,||y||_p,$$

that is, $y\in M(E)$ and $||y||_{M(E)} \ \leq \ C_6\,||y||_p$.

The theorem is proved.

Particularly, $L_{pq}\,(p\leq q\leq\infty)$ is interpolation space between L_p and $L_{p\infty}$ [12, p.142]. Therefore, we obtain the following

Corollary 1.12. If $1<p<\infty, p\leq q\leq\infty$, then

$$M(\,L_{pq}\,) \ = \ L_p.$$

Remark 1.13. In case $q=\infty$ the last statement was proved in [11].

Let us introduce a quantity connected with the boundedness of tensor product in the symmetric spaces. We define for a symmetric space $E,\ m\in\mathbf{N}$ and $\alpha=(\alpha_k)_{k=1}^m\in\mathbf{R^m}$

$$\mathcal{K}_E^m(\alpha) \ = \ \sup||\sum\nolimits_{k=1}^m \alpha_k y_k||_E,$$

where the supremum is taken over all $y\geq 0$, $||y||_E=1$ and all collections of disjointly supported functions y_k, such that

$$n_{y_k}(s) \ = \ \frac{1}{m}\,n_y(s),\ s>0.$$

Theorem 1.14. Let E be a symmetric space on I with an order subcontinuous norm. Then B is the bounded operator from $E\times E$ into $E(I\times I)$ if and only if there exists $C>0$ such that for all $m\in\mathbf{N},\alpha=(\alpha_i)_{i=1}^m\in\mathbf{R}^m$

$$\mathcal{K}_E^m(\alpha) \ \leq \ C||\sum\nolimits_{i=1}^m \alpha_i\chi_{(\frac{i-1}{m},\frac{i}{m}]}||_E \tag{14}$$

Proof. If $y\in E, ||y||_E=1$ and $x=x(s)$ is such as in (9),then functions

$$B(x,y)(s,t) \ = \ \sum\nolimits_{i=1}^m \alpha_i\chi_{(\frac{i-1}{m},\frac{i}{m}]}(s)\,y(t) \ (s,t\in[0,1])$$

and

$$v(t) \;=\; \sum_{k=1}^{m} \alpha_k y_k(t)\;\; (t \in [0,1])$$

(y_k are mutually disjoint and $n_{y_k}(u) = \frac{1}{m} n_y(u)$, $u > 0$) are equimeasurable. Consequently, the inequality (14) is equivalent to the following:

$$||x \otimes y||_E \;\leq\; C\,||x||_E, \tag{15}$$

where $||y||_E = 1$ and the function x belongs to the class (9). Hence boundedness of the operator B from $E \times E$ into $E(I \times I)$ implies (14).

Conversely, let us assume that the inequality (14) (or (15)) is satisfied. Since the norm of E is order subcontinuous, then the inequality (15) can be extended, at first, on countably-value and, next, on arbitrary functions from E as in proof of the theorem 1.5.

2. *The tensor product in the Lorentz spaces* L_{pq}

Let $1 < p < \infty$, $1 \leq q \leq \infty$, $\alpha \in \mathbf{R}$, $\phi_{p,\alpha}(u) = u^{1/p} \ln^{\alpha}(e/u)$. The Lorentz-Zygmund space $L_{pq}(\log L)^{\alpha}$ consists of all measurable functions on I (or $I \times I$) for which

$$||x||_{p,\alpha,q} \;=\; \begin{cases} \left\{ \int\limits_0^1 (x^*(u)\phi_{p,\alpha}(u))^q \, du/u \right\}^{1/q}, & \text{if } q < \infty \\ \sup\limits_{0<u\leq 1} (x^*(u)\phi_{p,\alpha}(u)), & \text{if } q = \infty \end{cases}$$

is finite. As in case L_{pq} $||x||_{p,\alpha,q}$ is not a norm. However using the Minkovski inequality (see, for example, [16, p. 38]) it can be verified that this functional is equivalent to the norm $||x||'_{p,\alpha,q} = ||x^{**}||_{p,\alpha,q}$. Moreover, the fundamental function of space $L_{pq}(\log L)^{\alpha}$ is equivalent to the function $\phi_{p,\alpha}(u)$. For $\alpha = 0$ the Lorentz-Zygmund spaces are the standard Lorentz spaces L_{pq}, while for $p = q < \infty$ they are the Orlicz spaces $L_p(\log L)^{\alpha p}$. For $p = q = \infty$, they produce the exponential classes of Zygmund (see [17] for further details). The Lorentz-Zygmund spaces were studied of a number of authors (except [17] see, for example, [18]).

If E is a symmetric space, then we use E^0 to denote the norm closure of L_∞ in E.

Theorem 2.1. Let $1 < p < \infty$, $p \leq r \leq q \leq \infty$, $\alpha = r^{-1} - p^{-1}$. The operator B is bounded operator from $L_{pr} \times L_{pq}$ into the space $(L_{pq}(\log L)^{\alpha})^0\ (I \times I)$ (if $p < r < q = \infty$) and into $L_{pq}(\log L)^{\alpha}\ (I \times I)$ (otherwise).

We need for the proof of this theorem two lemmas. The first of them is proved immediately using straightforward calculations.

Lemma 2.2. Let $1 < p < \infty$, $\alpha \in \mathbf{R}$, $\psi_{p,\alpha} = 1/\phi_{p,\alpha}$.

1) If $C = C(p,\alpha) > 0$ is large enough, then for $v > C$ the distribution function of $\psi_{p\alpha}$ is equivalent to the function $N_{p,\alpha}(v) = v^{-p} \ln^{-p\alpha} v$.

2) If $\alpha_i < 1/p$ $(i = 0,1)$, then the functions $\psi_{p,\alpha_0} \otimes \psi_{p,\alpha_1}$ and $\psi_{p,\alpha_0+\alpha_1-1/p}$ are equimeasurable.

The second lemma can be proved using standard interpolation techniques (we shall denote by $[X_0, X_1]_\theta$ the complex interpolation spaces, as defined, for example, in [13] or [19]). In detail see [11].

Lemma 2.3. Let $1 < p_1 \le p_0 < \infty$, $-\infty < \alpha_1 \le \alpha_0 < \infty$, $1 \le q_0 \le q_1 \le \infty$, $0 \le \theta \le 1$ and

$$\max (p_0 - p_1 \, , \, \alpha_0 - \alpha_1 \, , \, q_1 - q_0) \; > \; 0 \tag{16}$$

Then the space

$$[\, L_{p_0,q_0}(\log L)^{\alpha_0} \; , \; L_{p_1,q_1}(\log L)^{\alpha_1} \,]_\theta$$

is isomorphic to $(\, L_{pq}(\log L)^{\alpha} \,)^0$, if $q_0 = q_1 = \infty$ and $0 < \theta \le 1$ or $q_1 = \infty$ and $\theta = 1$, and to $L_{pq}(\log L)^{\alpha}$, otherwise. The numbers p, q, α are defined by the following way:

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \, \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \, \alpha \, = \, (1-\theta)\alpha_0 \, + \, \theta\alpha_1.$$

Remark 2.4. The inequality (16) implies the embedding

$$L_{p_0,q_0}(\log L)^{\alpha_0} \; \subset \; L_{p_1,q_1}(\log L)^{\alpha_1}$$

that is used by proof of lemma 2.3. However this embedding is true even when (16) is not satisfied [17].

Proof of the theorem 2.1. Let us define numbers

$$s \, = \, \frac{pq}{r} \, , \, \theta \, = \, 1 - \frac{p}{r} \tag{17}$$

Then $p \le s \le \infty, 0 \le \theta \le 1, p \le r \le \infty, s \le q \le \infty$ and statement of theorem is true for extreme values r and q, that is,

$$B \, : \, L_{pp} \times L_{ps} \; \to \; L_{ps}(I \times I) \tag{18}$$

$$B \, : \, L_{p\infty} \times L_{p\infty} \; \to \; L_{p\infty}(\log L)^{-1/p} \tag{19}$$

In fact, (18) follows from the O'Neil theorem (or from the corollary 1.12 of present paper). In view of the Closed Graph Theorem for proof (19) it is enough to show that $B(x, y)$ belongs to $L_{p\infty}(\log L)^{-1/p} \, (I \times I)$, if $x \in L_{p\infty}, y \in L_{p\infty}$. Lemma 2.2 and definition of $L_{p\infty}(\log L)^{-1/p}$ imply that this is true for the functions $x_0(u) \, = \, u^{-1/p}$, $y_0(v) \, = \, v^{-1/p}$. Since these functions have the greatest rearrangement in $L_{p\infty}$, then (19) is proved.

For θ from (17) we use now the complex method of interpolation. At first, we remind its following property (see [19] or [13, p. 125-126]).

If (X_0, Y_0) , (Y_0, Y_1) and (Z_0, Z_1) are arbitrary Banach pairs, T is an bilinear operator from $X_i \times Y_i$ into $Z_i \, (i = 0, 1)$, then T is bounded operator from $[X_0, X_1]_\theta \times [Y_0, Y_1]_\theta$ into $[Z_0, Z_1]_\theta \, (0 \le \theta \le 1)$.

The statement of theorem 2.1 follows now from last remarks and lemma 2.3.

For proof of following result about the exactness of theorem 2.1. it is sufficient to consider operator B on the pairs of functions of type

$$\psi_{p\alpha}(u) \, = \, u^{-1/p} ln^{-\alpha}(e/u)$$

and to use, next, lemma 2.2.

Theorem 2.5. If $1 < p < \infty$ and $p < r \leq q \leq \infty$,then the operator B does not act from $L_{pr} \times L_{pq}$ into the space $L_{pq}(\log L)^{\beta}\,(I \times I)$ for any $\beta > r^{-1} - p^{-1}$.

Remark 2.6. If $r = q = \infty$, then the following assertion is true: $L_{p\infty}(\log L)^{-1/p}$ is the smallest among symmetric spaces E such that B is the bounded operator from $L_{p\infty} \times L_{p\infty}$ into $E(I \times I)$ (see lemma 2.2 and proof of theorem 2.1).

We get from O'Neil theorem and theorem 2.2 following

Corollary 2.7. Let $1 < p \leq r \leq q < \infty, r^{-1} + q^{-1} - p^{-1} \geq 0$. If $\alpha = r^{-1} - p^{-1}$, $s^{-1} = r^{-1} + q^{-1} - p^{-1}$, then B is the bounded operator from $L_{pr} \times L_{pq}$ into the space

$$E(p, r, q)(I \times I) \; = \; L_{ps} \bigcap L_{pq}(\log L)^{\alpha}\,(I \times I).$$

Remark 2.8. The space $E(p, q, r)$ is not equal neither L_{ps} nor $L_{pq}(\log L)^{\alpha}$, if $p < r$. Assume, for example, that $r = q = 2p$. Then B acts from $L_{p,2p} \times L_{p,2p}$ into $E(p, 2p, 2p)(I \times I) = L_{p\infty} \bigcap L_{p,2p}(\log L)^{-1/2p}\,(I \times I)$.

On the one hand, it is easy to sheck that the function $\psi_{p,0}(u) = u^{-1/p}$ belongs to $L_{p\infty}$ and it does not belong to $L_{p,2p}(\log L)^{-1/2p}$.

On the other hand, if we assume that, vice versa,

$$L_{p,2p}(\log L)^{-1/2p} \subset L_{p\infty},$$

then from the inequality for fundamental functions of these spaces we obtain false inequality: $\ln(e/t) \leq C$ ($C > 0$ does not depend on $t \in (0, 1]$).

Remark 2.9. As we remarked in beginning of this paper the case

$$\frac{1}{r} + \frac{1}{q} - \frac{1}{p} < 0$$

is not considered in the O'Neil theorem. The theorems 2.1 and 2.4 complete it, characterising the image $B(L_{pr}, L_{pq})$ in this case to within any positive power of logarithm.

References

1. M.Milman, Some new function spaces and their tensor products, Notes de Mat., 20 (1978) 1-128.

2. J.Lindenstrauss and L.Tzafriri, Classical Banach spaces 2, Function spaces, Springer, Berlin 1979.

3. W.B.Johnson, B.Maurey, G.Schechtman and L.Tzafriri, Symmetric structures in Banach spaces, Mem. AMS 217 (1979) 1-298.

4. N.L.Carothers, Rearrangement invariant subspaces of Lorentz function spaces, Isr. J. Math. 40, No.3-4 (1981) 217-228.

5. C.Corduneanu, Integral equations and stability of Feedback systems, Academic Press, New York 1973.

6. M.Milman, Tensor products of function spaces, Amer. Math. Soc. 82, No.4 (1976) 626-628.

7. M.Milman, Embeddings of Lorentz-Marcinkiewicz spaces with mixed norms, Anal. math. 4, No.3 (1978) 215-223.

8. M.Milman, Embeddings of $L(p, q)$ spaces and Orlicz spaces with mixed norms, Notes de Mat. 13 (1977) 1-7.

9. R.O'Neil, Integral transforms and tensor products on Orlicz spaces and $L(p, q)$ spaces, J. d'Analyse Math. 21 (1968) 1-276.

10. R.O'Neil, Convolution operators and $L(p, q)$ spaces, Duke Math. J. 30 (1963) 129-142.

11. S.V.Astashkin, On bilinear multiplicative operator [in Russian], In: Theory of functions of several variables, Jaroslavl' (1982) 3-15.

12. S.G.Krein, Yu.I.Petunin and E.M.Semenov, Interpolation of Linear Operators [in Russian], Nauka, Moskow 1978.

13. J.Bergh and J.Lofstrom, Interpolation Spaces. An Introduction [Russian translation], Mir, Moscow 1980.

14. H.P.Rosenthal, On a theorem of J.L.Krivine concerning block finite-representablity of l^p in general Banach spaces, J. Funct. Anal. 28 (1978) 197-225.

15. L.V.Kantorovicz and G.P.Akilov, Functional Analysis [in Russian], Nauka, Moscow 1977.

16. A.Zygmund, Trigonometric Series.V.1 [Russian translation], Mir, Moskow 1965.

17. R.Sharpley, Counterexamples for classical operators on Lorentz-Zygmund spaces, Studia Math. 68, No.2 (1980) 141-158.

18. B. Jawerth and M.Milman, Extrapolation theory with application, Mem. AMS 89,No.440 (1991) 1-82.

19. A.P.Calderon, Intermediate spaces and interpolation, the complex method, Studia Math. 24 (1964) 113-190.